

REPRESENTATIONS BY QUATERNARY QUADRATIC FORMS WITH COEFFICIENTS 1, 2, 5 OR 10

AYŞE ALACA AND MADA ALTIARY

ABSTRACT. We determine explicit formulas for the number of representations of a positive integer n by quaternary quadratic forms with coefficients 1, 2, 5 or 10. We use a modular forms approach.

Key words and phrases: quaternary quadratic forms, representations, theta functions, Dedekind eta function, eta quotients, modular forms, Eisenstein forms, cusp forms.

2010 Mathematics Subject Classification: 11E25, 11E20, 11F11, 11F20, 11F27.

1. INTRODUCTION

Let \mathbb{N} , \mathbb{N}_0 , \mathbb{Z} and \mathbb{C} denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. For $n \in \mathbb{N}$ we set $\sigma(n) = \sum_{d|n} d$, where d runs through the positive divisors of n . If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$. For $a_1, a_2, a_3, a_4 \in \mathbb{N}$, and $n \in \mathbb{N}_0$ we define

$$N(a_1, a_2, a_3, a_4; n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}.$$

It is a classical result of Jacobi [7, 21] that

$$N(1, 1, 1, 1; n) = 8\sigma(n) - 32\sigma(n/4).$$

Formulas for $N(a_1, a_2, a_3, a_4; n)$ for the quaternary quadratic forms

$$(a_1, a_2, a_3, a_4) = (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2), (1, 1, 1, 5), (1, 1, 5, 5), (1, 5, 5, 5)$$

are in the literature, see for example [1, 2, 3, 8, 12, 13, 14, 15, 16, 17, 20].

There are twenty-six quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 5, 10\}$, $a_1 \leq a_2 \leq a_3 \leq a_4$ and $\gcd(a_1, a_2, a_3, a_4) = 1$, see Table 2.1. In this paper, we determine an explicit formula for $N(a_1, a_2, a_3, a_4; n)$ for each of these quaternary forms in a uniform manner. We use a modular forms approach.

For $q \in \mathbb{C}$ with $|q| < 1$, Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

For $a_1, a_2, a_3, a_4 \in \mathbb{N}$ we have

$$(1.1) \quad \sum_{n=1}^{\infty} N(a_1, a_2, a_3, a_4; n) q^n = \varphi(q^{a_1}) \varphi(q^{a_2}) \varphi(q^{a_3}) \varphi(q^{a_4}).$$

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ by

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

Throughout the remainder of the paper we take $q = q(z) := e^{2\pi iz}$ with $z \in \mathbb{H}$. Thus we can express $\eta(z)$ as

$$(1.2) \quad \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{\delta} \eta^{r_{\delta}}(\delta z),$$

where δ runs through a finite set of positive integers and the exponents r_{δ} are non-zero integers. It is known (see for example [5, Corollary 1.3.4]) that

$$(1.3) \quad \varphi(q) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}.$$

2. MODULAR SPACES $M_2(\Gamma_0(40), \chi_i)$ WITH $i \in \{0, 1, 2, 3\}$

For $n \in \mathbb{N}$ and Dirichlet characters χ and ψ we define $\sigma_{\chi, \psi}(n)$ by

$$(2.1) \quad \sigma_{\chi, \psi}(n) := \sum_{1 \leq m \mid n} \psi(m) \chi(n/m) m.$$

If $n \notin \mathbb{N}$ we set $\sigma_{\chi, \psi}(n) = 0$. Let χ_0 denote the trivial character, that is $\chi_0(n) = 1$ for all $n \in \mathbb{Z}$. Hence $\sigma_{\chi_0, \chi_0}(n)$ coincides with the sum of divisors function $\sigma(n)$. Let $N \in \mathbb{N}$. The modular subgroup $\Gamma_0(N)$ is defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1, \ c \equiv 0 \pmod{N} \right\}.$$

Let χ be a Dirichlet character of modulus dividing N and let $k \in \mathbb{Z}$. We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. If $\chi = \chi_0$, then we write $M_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi_0)$, and $S_k(\Gamma_0(N))$ for $S_k(\Gamma_0(N), \chi_0)$. It is known (see for example [19, p. 83]) that

$$(2.2) \quad M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi).$$

For $n \in \mathbb{Z}$ we define three Dirichlet characters by

$$(2.3) \quad \chi_1(n) = \left(\frac{5}{n}\right), \quad \chi_2(n) = \left(\frac{8}{n}\right), \quad \chi_3(n) = \left(\frac{40}{n}\right).$$

We define the Eisenstein series

$$(2.4) \quad L(q) := E_{\chi_0, \chi_0}(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n,$$

$$(2.5) \quad E_{\chi_0, \chi_1}(q) = -\frac{1}{5} + \sum_{n=1}^{\infty} \sigma_{\chi_0, \chi_1}(n)q^n, \quad E_{\chi_1, \chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_1, \chi_0}(n)q^n,$$

$$(2.6) \quad E_{\chi_0, \chi_2}(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \sigma_{\chi_0, \chi_2}(n)q^n, \quad E_{\chi_2, \chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_2, \chi_0}(n)q^n,$$

$$(2.7) \quad E_{\chi_0, \chi_3}(q) = -7 + \sum_{n=1}^{\infty} \sigma_{\chi_0, \chi_3}(n)q^n, \quad E_{\chi_3, \chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_3, \chi_0}(n)q^n,$$

$$(2.8) \quad E_{\chi_1, \chi_2}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_1, \chi_2}(n)q^n, \quad E_{\chi_2, \chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{\chi_2, \chi_1}(n)q^n.$$

We use the following lemma to determine if certain eta quotients are modular forms. See [6, p. 174], [10, Corollary 2.3, p. 37], [9, Theorem 5.7, p. 101], [11] and [18, Theorem 1.64].

Lemma 2.1. (Ligozat) *Let $N \in \mathbb{N}$ and $f(z) = \prod_{1 \leq \delta | N} \eta^{r_\delta}(\delta z)$ be an eta quotient.*

Let $s = \prod_{1 \leq \delta | N} \delta^{|r_\delta|}$ and $k = \frac{1}{2} \sum_{1 \leq \delta | N} r_\delta$. Suppose that the following conditions are satisfied:

$$(L1) \quad \sum_{1 \leq \delta | N} \delta \cdot r_\delta \equiv 0 \pmod{24},$$

$$(L2) \quad \sum_{1 \leq \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24},$$

$$(L3) \quad \sum_{1 \leq \delta | N} \frac{\gcd(d, \delta)^2 \cdot r_\delta}{\delta} \geq 0 \text{ for each positive divisor } d \text{ of } N,$$

$$(L4) \quad k \text{ is an integer.}$$

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, where the character χ is given by $\chi(m) = \left(\frac{(-1)^k s}{m}\right)$.

(L3)' In addition to the above conditions, if the inequality in (L3) is strict for each positive divisor d of N , then $f(z) \in S_k(\Gamma_0(N), \chi)$.

In Table 2.1, we group our twenty-six quaternary forms (a_1, a_2, a_3, a_4) according to modular spaces $M_2(\Gamma_0(40), \chi)$ to which $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ belong.

Table 2.1

$M_2(\Gamma_0(40))$	$M_2(\Gamma_0(40), \chi_1)$	$M_2(\Gamma_0(40), \chi_2)$	$M_2(\Gamma_0(40), \chi_3)$
$(1, 1, 1, 1)\checkmark$	$(1, 1, 1, 5)\checkmark$	$(1, 1, 1, 2)\checkmark$	$(1, 1, 1, 10)$
$(1, 1, 2, 2)\checkmark$	$(1, 1, 2, 10)*$	$(1, 1, 5, 10)$	$(1, 1, 2, 5)*$
$(1, 1, 5, 5)\checkmark$	$(1, 2, 2, 5)*$	$(1, 2, 2, 2)\checkmark$	$(1, 2, 2, 10)$
$(1, 1, 10, 10)$	$(1, 5, 5, 5)\checkmark$	$(1, 2, 5, 5)$	$(1, 5, 5, 10)$
$(1, 2, 5, 10)*$	$(1, 5, 10, 10)$	$(1, 2, 10, 10)$	$(1, 10, 10, 10)$
$(2, 2, 5, 5)$	$(2, 5, 5, 10)$	$(2, 2, 5, 10)$	$(2, 2, 2, 5)$
			$(2, 5, 5, 5)$
			$(2, 5, 10, 10)$

Formulas $N(a_1, a_2, a_3, a_4; n)$ for the forms with a checkmark (\checkmark) in Table 2.1 are known. Of the remaining nineteen forms, four are universal and identified with an asterisk (*).

We deduce from [19, Sec. 6.1, p. 93] that

$$(2.9) \quad \dim(E_2(\Gamma_0(40))) = 7, \dim(S_2(\Gamma_0(40))) = 3.$$

We also deduce from [19, Sec. 6.3, p. 98] that

$$(2.10) \quad \dim(E_2(\Gamma_0(40), \chi_1)) = 8, \dim(S_2(\Gamma_0(40), \chi_1)) = 2,$$

$$(2.11) \quad \dim(E_2(\Gamma_0(40), \chi_2)) = 4, \dim(S_2(\Gamma_0(40), \chi_2)) = 4,$$

$$(2.12) \quad \dim(E_2(\Gamma_0(40), \chi_3)) = 4, \dim(S_2(\Gamma_0(40), \chi_3)) = 4.$$

Theorem 2.1. *Let χ_1, χ_2, χ_3 be as in (2.3). If (a_1, a_2, a_3, a_4) is in the first, second, third or fourth column of Table 2.1, then*

$$\begin{aligned} \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) &\in M_2(\Gamma_0(40)), \\ \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) &\in M_2(\Gamma_0(40), \chi_1), \\ \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) &\in M_2(\Gamma_0(40), \chi_2), \\ \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) &\in M_2(\Gamma_0(40), \chi_3), \end{aligned}$$

respectively.

Proof. The assertion directly follows from (1.3) and Lemma 2.1. ■

Let $n \in \mathbb{N}$. We define the eta quotients $A_k(q)$, $B_k(q)$, $C_k(q)$, $D_k(q)$ and integers $a_k(n)$, $b_k(n)$, $c_k(n)$, $d_k(n)$ as follows:

$$(2.13) \quad A_1(q) = \sum_{n=1}^{\infty} a_1(n)q^n = \eta^2(2z)\eta^2(10z),$$

$$(2.14) \quad A_2(q) = A_1(q^2) = \sum_{n=1}^{\infty} a_2(n)q^n = \eta^2(4z)\eta^2(20z),$$

$$(2.15) \quad A_3(q) = \sum_{n=1}^{\infty} a_3(n)q^n = \frac{\eta^5(4z)\eta(10z)\eta^2(40z)}{\eta(2z)\eta^2(8z)\eta(20z)},$$

$$(2.16) \quad B_1(q) = \sum_{n=1}^{\infty} b_1(n)q^n = \frac{\eta(2z)\eta^4(20z)}{\eta(10z)},$$

$$(2.17) \quad B_2(q) = \sum_{n=1}^{\infty} b_2(n)q^n = \frac{\eta^4(4z)\eta(10z)}{\eta(2z)},$$

$$(2.18) \quad C_1(q) = \sum_{n=1}^{\infty} c_1(n)q^n = \frac{\eta^2(z)\eta(8z)\eta^2(10z)\eta(40z)}{\eta(2z)\eta(20z)},$$

$$(2.19) \quad C_2(q) = \sum_{n=1}^{\infty} c_2(n)q^n = \frac{\eta(z)\eta(5z)\eta^2(8z)\eta^2(20z)}{\eta(4z)\eta(10z)},$$

$$(2.20) \quad C_3(q) = \sum_{n=1}^{\infty} c_3(n)q^n = \frac{\eta^6(2z)\eta(10z)\eta^2(40z)}{\eta^2(z)\eta^2(4z)\eta(20z)},$$

$$(2.21) \quad C_4(q) = \sum_{n=1}^{\infty} c_4(n)q^n = \frac{\eta^6(4z)\eta^2(5z)\eta(20z)}{\eta^2(2z)\eta^2(8z)\eta(10z)},$$

$$(2.22) \quad D_1(q) = \sum_{n=1}^{\infty} d_1(n)q^n = \frac{\eta^2(z)\eta^6(4z)\eta(20z)}{\eta^3(2z)\eta^2(8z)},$$

$$(2.23) \quad D_2(q) = \sum_{n=1}^{\infty} d_2(n)q^n = \frac{\eta^2(5z)\eta(8z)\eta(10z)\eta(40z)}{\eta(20z)},$$

$$(2.24) \quad D_3(q) = \sum_{n=1}^{\infty} d_3(n)q^n = \frac{\eta(z)\eta(5z)\eta(20z)\eta^2(40z)}{\eta(10z)},$$

$$(2.25) \quad D_4(q) = \sum_{n=1}^{\infty} d_4(n)q^n = \frac{\eta(z)\eta(4z)\eta(5z)\eta^2(8z)}{\eta(2z)}.$$

Theorem 2.2. *Let χ_1, χ_2, χ_3 be as in (2.3). Then*

$$\{A_1(q), A_2(q), A_3(q)\}, \quad \{B_1(q), B_2(q)\}, \\ \{C_1(q), C_2(q), C_3(q), C_4(q)\}, \quad \{D_1(q), D_2(q), D_3(q), D_4(q)\}$$

are bases for $S_2(\Gamma_0(40))$, $S_2(\Gamma_0(40), \chi_1)$, $S_2(\Gamma_0(40), \chi_2)$ and $S_2(\Gamma_0(40), \chi_3)$, respectively.

Proof. The set $\{A_1(q), A_2(q), A_3(q)\}$ is linearly independent over \mathbb{C} . By Lemma 2.1, we have $A_k(q) \in S_2(\Gamma_0(40))$ for $k = 1, 2, 3$. The assertion now follows from (2.9). Similarly, the remaining three assertions follow from (2.10), (2.11), (2.12) and Lemma 2.1. ■

Theorem 2.3. *Let χ_0 be the trivial character and χ_1, χ_2, χ_3 be as in (2.3). Then*

$$\begin{aligned} &\{L(q) - tL(q^t) \mid t = 2, 4, 5, 8, 10, 20, 40\}, \\ &\{E_{\chi_0, \chi_1}(q^t), E_{\chi_1, \chi_0}(q^t) \mid t = 1, 2, 4, 8\}, \\ &\{E_{\chi_0, \chi_2}(q^t), E_{\chi_2, \chi_0}(q^t) \mid t = 1, 5\}, \\ &\{E_{\chi_0, \chi_3}(q), E_{\chi_1, \chi_2}(q), E_{\chi_2, \chi_1}(q), E_{\chi_3, \chi_0}(q)\} \end{aligned}$$

are bases for $E_2(\Gamma_0(40))$, $E_2(\Gamma_0(40), \chi_1)$, $E_2(\Gamma_0(40), \chi_2)$ and $E_2(\Gamma_0(40), \chi_3)$, respectively.

Proof. The assertions follow from [19, Theorem 5.9] with $\chi = \psi = \chi_0$; $\epsilon = \chi_1$ and $\chi, \psi \in \{\chi_0, \chi_1\}$; $\epsilon = \chi_2$ and $\chi, \psi \in \{\chi_0, \chi_2\}$; $\epsilon = \chi_3$ and $\chi, \psi \in \{\chi_0, \chi_1, \chi_2, \chi_3\}$, respectively. ■

Theorem 2.4. *Let χ_0 be the trivial character and χ_1, χ_2, χ_3 be as in (2.3). Then*

$$\begin{aligned} &\{L(q) - tL(q^t) \mid t = 2, 4, 5, 8, 10, 20, 40\} \cup \{A_1(q), A_2(q), A_3(q)\}, \\ &\{E_{\chi_0, \chi_1}(q^t), E_{\chi_1, \chi_0}(q^t) \mid t = 1, 2, 4, 8\} \cup \{B_1(q), B_2(q)\}, \\ &\{E_{\chi_0, \chi_2}(q^t), E_{\chi_2, \chi_0}(q^t) \mid t = 1, 5\} \cup \{C_k(q) \mid k = 1, 2, 3, 4\}, \\ &\{E_{\chi_0, \chi_3}(q), E_{\chi_1, \chi_2}(q), E_{\chi_2, \chi_1}(q), E_{\chi_3, \chi_0}(q)\} \cup \{D_k(q) \mid k = 1, 2, 3, 4\} \end{aligned}$$

are bases for $M_2(\Gamma_0(40))$, $M_2(\Gamma_0(40), \chi_1)$, $M_2(\Gamma_0(40), \chi_2)$, $M_2(\Gamma_0(40), \chi_3)$, respectively.

Proof. The assertions follow from (2.2), Theorems 2.2 and 2.3. ■

We now give four theorems (Theorems 2.5–2.8) from which the theorems of Section 3 (Theorems 3.1–3.4) follow.

Theorem 2.5. *We have*

$$\begin{aligned} \varphi^4(q) &= 8L(q) - 32L(q^4), \\ \varphi^2(q)\varphi^2(q^2) &= 4L(q) - 4L(q^2) + 8L(q^4) - 32L(q^8), \\ \varphi^2(q)\varphi^2(q^5) &= \frac{4}{3}L(q) - \frac{16}{3}L(q^4) + \frac{20}{3}L(q^5) - \frac{80}{3}L(q^{20}) + \frac{8}{3}A_1(q), \\ \varphi^2(q)\varphi^2(q^{10}) &= \frac{2}{3}L(q) - \frac{2}{3}L(q^2) + \frac{4}{3}L(q^4) + \frac{10}{3}L(q^5) - \frac{16}{3}L(q^8) - \frac{10}{3}L(q^{10}) \\ &\quad + \frac{20}{3}L(q^{20}) - \frac{80}{3}L(q^{40}) + \frac{10}{3}A_1(q) + \frac{8}{3}A_2(q) + 4A_3(q), \\ \varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10}) &= L(q) - L(q^2) - 2L(q^4) - 5L(q^5) + 8L(q^8) \\ &\quad + 5L(q^{10}) + 10L(q^{20}) - 40L(q^{40}) + A_1(q) + 2A_3(q), \\ \varphi^2(q^2)\varphi^2(q^5) &= \frac{2}{3}L(q) - \frac{2}{3}L(q^2) + \frac{4}{3}L(q^4) + \frac{10}{3}L(q^5) - \frac{16}{3}L(q^8) - \frac{10}{3}L(q^{10}) \\ &\quad + \frac{20}{3}L(q^{20}) - \frac{80}{3}L(q^{40}) - \frac{2}{3}A_1(q) + \frac{8}{3}A_2(q) - 4A_3(q). \end{aligned}$$

Proof. Let (a_1, a_2, a_3, a_4) be any of the quaternary quadratic forms listed in the first column of Table 2.1. By Theorem 2.1 we have $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40))$. By Theorem 2.4, $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $L(q) - tL(q^t)$ ($t = 2, 4, 5, 8, 10, 20, 40$) and $A_k(q)$ ($k \in \{1, 2, 3\}$), namely

$$\begin{aligned}
 \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) = & x_1(L(q) - 2L(q^2)) + x_2(L(q) - 4L(q^4)) \\
 & + x_3(L(q) - 5L(q^5)) + x_4(L(q) - 8L(q^8)) \\
 (2.26) \quad & + x_5(L(q) - 10L(q^{10})) + x_6(L(q) - 20L(q^{20})) \\
 & + x_7(L(q) - 40L(q^{40})) + y_1A_1(q) + y_2A_2(q) + y_3A_3(q).
 \end{aligned}$$

We only prove the last equation in the theorem as the others can be proven similarly. Let $(a_1, a_2, a_3, a_4) = (2, 2, 5, 5)$. Appealing to [9, Theorem 3.13], we find that the Sturm bound for the modular space $M_2(\Gamma_0(40))$ is 12. So, equating the coefficients of q^n for $0 \leq n \leq 12$ on both sides of (2.26), we find a system of linear equations with the unknowns x_i ($1 \leq i \leq 7$), y_1 , y_2 and y_3 . Using MAPLE we solve the system and find that

$$x_1 = x_5 = \frac{1}{3}, x_2 = x_6 = -\frac{1}{3}, x_3 = y_1 = -\frac{2}{3}, x_4 = x_7 = \frac{2}{3}, y_2 = \frac{8}{3}, y_3 = -4.$$

Substituting these values back in (2.26), and with the obvious simplifications, we find the asserted equation. \blacksquare

Corollary 2.1. *Let $n \in \mathbb{N}$. We have*

$$N(1, 1, 10, 10; n) = N(2, 2, 5, 5; n) \text{ if } n \equiv 0 \pmod{2}.$$

Proof. From Theorem 2.5, we have

$$(2.27) \quad \varphi^2(q)\varphi^2(q^{10}) - \varphi^2(q^2)\varphi^2(q^5) = 4A_1(q) + 8A_3(q).$$

It is clear from (1.2), (2.13) and (2.15) that

$$(2.28) \quad a_1(n) = a_3(n) = 0 \text{ if } n \equiv 0 \pmod{2}.$$

The assertion now follows from (1.1), (2.27) and (2.28). \blacksquare

Similarly to Theorem 2.5, Theorems 2.6–2.8 follow from Theorems 2.1 and 2.4.

Theorem 2.6. *Let χ_0 be the trivial character and χ_1 be as in (2.3). Then*

$$\begin{aligned}
 \varphi^3(q)\varphi(q^5) = & E_{\chi_0, \chi_1}(q) - 2E_{\chi_0, \chi_1}(q^2) - 4E_{\chi_0, \chi_1}(q^4) + 5E_{\chi_1, \chi_0}(q) \\
 & + 10E_{\chi_1, \chi_0}(q^2) - 20E_{\chi_1, \chi_0}(q^4), \\
 \varphi^2(q)\varphi(q^2)\varphi(q^{10}) = & -\frac{1}{2}E_{\chi_0, \chi_1}(q) + \frac{1}{2}E_{\chi_0, \chi_1}(q^2) - E_{\chi_0, \chi_1}(q^4) \\
 & - 4E_{\chi_0, \chi_1}(q^8) + \frac{5}{2}E_{\chi_1, \chi_0}(q) + \frac{5}{2}E_{\chi_1, \chi_0}(q^2) \\
 & + 5E_{\chi_1, \chi_0}(q^4) - 20E_{\chi_1, \chi_0}(q^8) + 2B_2(q),
 \end{aligned}$$

$$\begin{aligned}
\varphi(q)\varphi^2(q^2)\varphi(q^5) &= \frac{1}{2}E_{\chi_0, \chi_1}(q) - \frac{1}{2}E_{\chi_0, \chi_1}(q^2) - E_{\chi_0, \chi_1}(q^4) \\
&\quad - 4E_{\chi_0, \chi_1}(q^8) + \frac{5}{2}E_{\chi_1, \chi_0}(q) + \frac{5}{2}E_{\chi_1, \chi_0}(q^2) \\
&\quad - 5E_{\chi_1, \chi_0}(q^4) + 20E_{\chi_1, \chi_0}(q^8) + 5B_1(q) - B_2(q), \\
\varphi(q)\varphi^3(q^5) &= E_{\chi_0, \chi_1}(q) - 2E_{\chi_0, \chi_1}(q^2) - 4E_{\chi_0, \chi_1}(q^4) + E_{\chi_1, \chi_0}(q) \\
&\quad + 2E_{\chi_1, \chi_0}(q^2) - 4E_{\chi_1, \chi_0}(q^4), \\
\varphi(q)\varphi(q^5)\varphi^2(q^{10}) &= \frac{1}{2}E_{\chi_0, \chi_1}(q) - \frac{1}{2}E_{\chi_0, \chi_1}(q^2) - E_{\chi_0, \chi_1}(q^4) \\
&\quad - 4E_{\chi_0, \chi_1}(q^8) + \frac{1}{2}E_{\chi_1, \chi_0}(q) + \frac{1}{2}E_{\chi_1, \chi_0}(q^2) \\
&\quad - E_{\chi_1, \chi_0}(q^4) + 4E_{\chi_1, \chi_0}(q^8) - B_1(q) + B_2(q), \\
\varphi(q^2)\varphi^2(q^5)\varphi(q^{10}) &= -\frac{1}{2}E_{\chi_0, \chi_1}(q) + \frac{1}{2}E_{\chi_0, \chi_1}(q^2) - E_{\chi_0, \chi_1}(q^4) \\
&\quad - 4E_{\chi_0, \chi_1}(q^8) + \frac{1}{2}E_{\chi_1, \chi_0}(q) + \frac{1}{2}E_{\chi_1, \chi_0}(q^2) \\
&\quad + E_{\chi_1, \chi_0}(q^4) - 4E_{\chi_1, \chi_0}(q^8) - 2B_1(q).
\end{aligned}$$

Theorem 2.7. *Let χ_0 be the trivial character and χ_2 be as in (2.3). Then*

$$\begin{aligned}
\varphi^3(q)\varphi(q^2) &= -2E_{\chi_0, \chi_2}(q) + 8E_{\chi_2, \chi_0}(q), \\
\varphi^2(q)\varphi(q^5)\varphi(q^{10}) &= \frac{2}{13}(2E_{\chi_0, \chi_2}(q) - 15E_{\chi_0, \chi_2}(q^5) + 8E_{\chi_2, \chi_0}(q) + 60E_{\chi_2, \chi_0}(q^5)) \\
&\quad + \frac{8}{13}(6C_1(q) - 4C_2(q) - 3C_3(q) + 4C_4(q)), \\
\varphi(q)\varphi^3(q^2) &= -2E_{\chi_0, \chi_2}(q) + 4E_{\chi_2, \chi_0}(q), \\
\varphi(q)\varphi(q^2)\varphi^2(q^5) &= \frac{2}{13}(-3E_{\chi_0, \chi_2}(q) - 10E_{\chi_0, \chi_2}(q^5) + 12E_{\chi_2, \chi_0}(q)) \\
&\quad - \frac{80}{13}E_{\chi_2, \chi_0}(q^5) + \frac{8}{13}(-2C_2(q) - 5C_3(q) + C_4(q)), \\
\varphi(q)\varphi(q^2)\varphi^2(q^{10}) &= \frac{2}{13}(-3E_{\chi_0, \chi_2}(q) - 10E_{\chi_0, \chi_2}(q^5) + 6E_{\chi_2, \chi_0}(q)) \\
&\quad - \frac{40}{13}E_{\chi_2, \chi_0}(q^5) + \frac{4}{13}(2C_1(q) - 2C_3(q) + 5C_4(q)), \\
\varphi^2(q^2)\varphi(q^5)\varphi(q^{10}) &= \frac{2}{13}(2E_{\chi_0, \chi_2}(q) - 15E_{\chi_0, \chi_2}(q^5) + 4E_{\chi_2, \chi_0}(q) + 30E_{\chi_2, \chi_0}(q^5)) \\
&\quad + \frac{4}{13}(-4C_1(q) + 12C_2(q) + 8C_3(q) - 3C_4(q)).
\end{aligned}$$

Theorem 2.8. *Let χ_0 be the trivial character and χ_1, χ_2, χ_3 be as in (2.2). Then*

$$\begin{aligned}
\varphi^3(q)\varphi(q^{10}) &= \frac{1}{7}(-E_{\chi_0, \chi_3}(q) - 5E_{\chi_1, \chi_2}(q) + 4E_{\chi_2, \chi_1}(q) + 20E_{\chi_3, \chi_0}(q)) \\
&\quad + \frac{4}{7}(-3D_1(q) + 15D_2(q) - 15D_3(q) + 9D_4(q)), \\
\varphi^2(q)\varphi(q^2)\varphi(q^5) &= \frac{1}{7}(-E_{\chi_0, \chi_3}(q) + 5E_{\chi_1, \chi_2}(q) - 4E_{\chi_2, \chi_1}(q) + 20E_{\chi_3, \chi_0}(q)) \\
&\quad + \frac{8}{7}(-D_1(q) + 2D_4(q)), \\
\varphi(q)\varphi^2(q^2)\varphi(q^{10}) &= \frac{1}{7}(-E_{\chi_0, \chi_3}(q) - 5E_{\chi_1, \chi_2}(q) + 2E_{\chi_2, \chi_1}(q) + 10E_{\chi_3, \chi_0}(q)) \\
&\quad + \frac{4}{7}(D_1(q) + 5D_2(q) + 5D_3(q) + D_4(q)), \\
\varphi(q)\varphi^2(q^5)\varphi(q^{10}) &= \frac{1}{7}(-E_{\chi_0, \chi_3}(q) - E_{\chi_1, \chi_2}(q) + 4E_{\chi_2, \chi_1}(q) + 4E_{\chi_3, \chi_0}(q)) \\
&\quad + \frac{8}{7}(D_2(q) - D_3(q) + D_4(q)), \\
\varphi(q)\varphi^3(q^{10}) &= -\frac{1}{7}(E_{\chi_0, \chi_3}(q) + E_{\chi_1, \chi_2}(q) - 2E_{\chi_2, \chi_1}(q) - 2E_{\chi_3, \chi_0}(q)) \\
&\quad + \frac{12}{7}(D_2(q) + D_3(q) + D_4(q)), \\
\varphi^3(q^2)\varphi(q^5) &= \frac{1}{7}(-E_{\chi_0, \chi_3}(q) + 5E_{\chi_1, \chi_2}(q) - 2E_{\chi_2, \chi_1}(q) + 10E_{\chi_3, \chi_0}(q) - 12D_1(q)), \\
\varphi(q^2)\varphi^3(q^5) &= \frac{1}{7}(-E_{\chi_0, \chi_3}(q) + E_{\chi_1, \chi_2}(q) - 4E_{\chi_2, \chi_1}(q) + 4E_{\chi_3, \chi_0}(q)) \\
&\quad - \frac{12}{7}(D_1(q) + D_2(q) + 3D_3(q) - D_4(q)), \\
\varphi(q^2)\varphi(q^5)\varphi^2(q^{10}) &= \frac{1}{7}(-E_{\chi_0, \chi_3}(q) + E_{\chi_1, \chi_2}(q) - 2E_{\chi_2, \chi_1}(q) + 2E_{\chi_3, \chi_0}(q)) \\
&\quad + \frac{4}{7}(-D_1(q) + D_2(q) - 3D_3(q) + D_4(q)).
\end{aligned}$$

3. MAIN RESULTS

Theorem 3.1. *Let $n \in \mathbb{N}$. We have*

$$N(1, 1, 5, 5; n) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) + \frac{8}{3}a_1(n),$$

$$\begin{aligned}
N(1, 1, 10, 10; n) &= \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8) \\
&\quad - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) + \frac{10}{3}a_1(n) \\
&\quad + \frac{8}{3}a_2(n) + 4a_3(n), \\
N(1, 2, 5, 10; n) &= \sigma(n) - \sigma(n/2) - 2\sigma(n/4) - 5\sigma(n/5) + 8\sigma(n/8) \\
&\quad + 5\sigma(n/10) + 10\sigma(n/20) - 40\sigma(n/40) + a_1(n) + 2a_3(n), \\
N(2, 2, 5, 5; n) &= \frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8) \\
&\quad - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) - \frac{2}{3}a_1(n) \\
&\quad + \frac{8}{3}a_2(n) - 4a_3(n).
\end{aligned}$$

Proof. The assertions follow from (1.1), (2.4) and Theorem 2.5. ■

Theorem 3.2. Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i, \chi_j}(n)$ be as in (2.1) for $i, j \in \{0, 1\}$. We have

$$\begin{aligned}
N(1, 1, 1, 5; n) &= \sigma_{\chi_0, \chi_1}(n) - 2\sigma_{\chi_0, \chi_1}(n/2) - 4\sigma_{\chi_0, \chi_1}(n/4) \\
&\quad + 5\sigma_{\chi_1, \chi_0}(n) + 10\sigma_{\chi_1, \chi_0}(n/2) - 20\sigma_{\chi_1, \chi_0}(n/4), \\
N(1, 1, 2, 10; n) &= -\frac{1}{2}\sigma_{\chi_0, \chi_1}(n) + \frac{1}{2}\sigma_{\chi_0, \chi_1}(n/2) - \sigma_{\chi_0, \chi_1}(n/4) - 4\sigma_{\chi_0, \chi_1}(n/8) \\
&\quad + \frac{5}{2}\sigma_{\chi_1, \chi_0}(n) + \frac{5}{2}\sigma_{\chi_1, \chi_0}(n/2) + 5\sigma_{\chi_1, \chi_0}(n/4) \\
&\quad - 20\sigma_{\chi_1, \chi_0}(n/8) + 2b_2(n), \\
N(1, 2, 2, 5; n) &= \frac{1}{2}\sigma_{\chi_0, \chi_1}(n) - \frac{1}{2}\sigma_{\chi_0, \chi_1}(n/2) - \sigma_{\chi_0, \chi_1}(n/4) - 4\sigma_{\chi_0, \chi_1}(n/8) \\
&\quad + \frac{5}{2}\sigma_{\chi_1, \chi_0}(n) + \frac{5}{2}\sigma_{\chi_1, \chi_0}(n/2) - 5\sigma_{\chi_1, \chi_0}(n/4) + 20\sigma_{\chi_1, \chi_0}(n/8) \\
&\quad + 5b_1(n) - b_2(n), \\
N(1, 5, 5, 5; n) &= \sigma_{\chi_0, \chi_1}(n) - 2\sigma_{\chi_0, \chi_1}(n/2) - 4\sigma_{\chi_0, \chi_1}(n/4) \\
&\quad + \sigma_{\chi_1, \chi_0}(n) + 2\sigma_{\chi_1, \chi_0}(n/2) - 4\sigma_{\chi_1, \chi_0}(n/4), \\
N(1, 5, 10, 10; n) &= \frac{1}{2}\sigma_{\chi_0, \chi_1}(n) - \frac{1}{2}\sigma_{\chi_0, \chi_1}(n/2) - \sigma_{\chi_0, \chi_1}(n/4) - 4\sigma_{\chi_0, \chi_1}(n/8) \\
&\quad + \frac{1}{2}\sigma_{\chi_1, \chi_0}(n) + \frac{1}{2}\sigma_{\chi_1, \chi_0}(n/2) - \sigma_{\chi_1, \chi_0}(n/4) + 4\sigma_{\chi_1, \chi_0}(n/8) \\
&\quad - b_1(n) + b_2(n),
\end{aligned}$$

$$\begin{aligned}
N(2, 5, 5, 10; n) = & -\frac{1}{2}\sigma_{\chi_0, \chi_1}(n) + \frac{1}{2}\sigma_{\chi_0, \chi_1}(n/2) - \sigma_{\chi_0, \chi_1}(n/4) - 4\sigma_{\chi_0, \chi_1}(n/8) \\
& + \frac{1}{2}\sigma_{\chi_1, \chi_0}(n) + \frac{1}{2}\sigma_{\chi_1, \chi_0}(n/2) + \sigma_{\chi_1, \chi_0}(n/4) \\
& - 4\sigma_{\chi_1, \chi_0}(n/8) - 2b_1(n).
\end{aligned}$$

Proof. The assertions follow from (1.1), (2.5) and Theorem 2.6. ■

Theorem 3.3. Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i, \chi_j}(n)$ be as in (2.1) for $i, j \in \{0, 2\}$. Then

$$\begin{aligned}
N(1, 1, 1, 2; n) &= -2\sigma_{\chi_0, \chi_2}(n) + 8\sigma_{\chi_2, \chi_0}(n), \\
N(1, 1, 5, 10; n) &= \frac{2}{13}(2\sigma_{\chi_0, \chi_2}(n) - 15\sigma_{\chi_0, \chi_2}(n/5) + 8\sigma_{\chi_2, \chi_0}(n) + 60\sigma_{\chi_2, \chi_0}(n/5)) \\
&\quad + \frac{8}{13}(6c_1(n) - 4c_2(n) - 3c_3(n) + 4c_4(n)), \\
N(1, 2, 2, 2; n) &= -2\sigma_{\chi_0, \chi_2}(n) + 4\sigma_{\chi_2, \chi_0}(n), \\
N(1, 2, 5, 5; n) &= -\frac{2}{13}(3\sigma_{\chi_0, \chi_2}(n) + 10\sigma_{\chi_0, \chi_2}(n/5) - 12\sigma_{\chi_2, \chi_0}(n) + 40\sigma_{\chi_2, \chi_0}(n/5)) \\
&\quad + \frac{8}{13}(-2c_2(n) - 5c_3(n) + c_4(n)), \\
N(1, 2, 10, 10; n) &= \frac{2}{13}(-3\sigma_{\chi_0, \chi_2}(n) - 10\sigma_{\chi_0, \chi_2}(n/5) + 6\sigma_{\chi_2, \chi_0}(n) - 20\sigma_{\chi_2, \chi_0}(n/5)) \\
&\quad + \frac{4}{13}(2c_1(n) - 2c_3(n) + 5c_4(n)), \\
N(2, 2, 5, 10; n) &= \frac{2}{13}(2\sigma_{\chi_0, \chi_2}(n) - 15\sigma_{\chi_0, \chi_2}(n/5) + 4\sigma_{\chi_2, \chi_0}(n) + 30\sigma_{\chi_2, \chi_0}(n/5)) \\
&\quad + \frac{4}{13}(-4c_1(n) + 12c_2(n) + 8c_3(n) - 3c_4(n)).
\end{aligned}$$

Proof. The assertions follow from (1.1), (2.6) and Theorem 2.7. ■

Theorem 3.4. Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i, \chi_j}(n)$ be as in (2.1) for $i, j \in \{0, 1, 2, 3\}$. Then

$$\begin{aligned}
N(1, 1, 1, 10; n) &= \frac{1}{7}(-\sigma_{\chi_0, \chi_3}(n) - 5\sigma_{\chi_1, \chi_2}(n) + 4\sigma_{\chi_2, \chi_1}(n) + 20\sigma_{\chi_3, \chi_0}(n)) \\
&\quad + \frac{4}{7}(-3d_1(n) + 15d_2(n) - 15d_3(n) + 9d_4(n)), \\
N(1, 1, 2, 5; n) &= \frac{1}{7}(-\sigma_{\chi_0, \chi_3}(n) + 5\sigma_{\chi_1, \chi_2}(n) - 4\sigma_{\chi_2, \chi_1}(n) + 20\sigma_{\chi_3, \chi_0}(n)) \\
&\quad + \frac{8}{7}(-d_1(n) + 2d_4(n)), \\
N(1, 2, 2, 10; n) &= \frac{1}{7}(-\sigma_{\chi_0, \chi_3}(n) - 5\sigma_{\chi_1, \chi_2}(n) + 2\sigma_{\chi_2, \chi_1}(n) + 10\sigma_{\chi_3, \chi_0}(n)) \\
&\quad + \frac{4}{7}(d_1(n) + 5d_2(n) + 5d_3(n) + d_4(n)),
\end{aligned}$$

$$\begin{aligned}
N(1, 5, 5, 10; n) &= \frac{1}{7} \left(-\sigma_{\chi_0, \chi_3}(n) - \sigma_{\chi_1, \chi_2}(n) + 4\sigma_{\chi_2, \chi_1}(n) + 4\sigma_{\chi_3, \chi_0}(n) \right) \\
&\quad + \frac{8}{7} (d_2(n) - d_3(n) + d_4(n)), \\
N(1, 10, 10, 10; n) &= \frac{1}{7} \left(-\sigma_{\chi_0, \chi_3}(n) - \sigma_{\chi_1, \chi_2}(n) + 2\sigma_{\chi_2, \chi_1}(n) + 2\sigma_{\chi_3, \chi_0}(n) \right) \\
&\quad + \frac{12}{7} (d_2(n) + d_3(n) + d_4(n)), \\
N(2, 2, 2, 5; n) &= \frac{1}{7} \left(-\sigma_{\chi_0, \chi_3}(n) + 5\sigma_{\chi_1, \chi_2}(n) - 2\sigma_{\chi_2, \chi_1}(n) + 10\sigma_{\chi_3, \chi_0}(n) \right) \\
&\quad - \frac{12}{7} d_1(n), \\
N(2, 5, 5, 5; n) &= \frac{1}{7} \left(-\sigma_{\chi_0, \chi_3}(n) + \sigma_{\chi_1, \chi_2}(n) - 4\sigma_{\chi_2, \chi_1}(n) + 4\sigma_{\chi_3, \chi_0}(n) \right) \\
&\quad + \frac{12}{7} (-d_1(n) - d_2(n) - 3d_3(n) + d_4(n)), \\
N(2, 5, 10, 10; n) &= \frac{1}{7} \left(-\sigma_{\chi_0, \chi_3}(n) + \sigma_{\chi_1, \chi_2}(n) - 2\sigma_{\chi_2, \chi_1}(n) + 2\sigma_{\chi_3, \chi_0}(n) \right) \\
&\quad + \frac{4}{7} (-d_1(n) + d_2(n) - 3d_3(n) + d_4(n)).
\end{aligned}$$

Proof. The assertions follow from (1.1), (2.7), (2.8) and Theorem 2.8. ■

4. REMARKS

Remark 4.1. Replacing q by $-q$ in $\varphi^3(q)\varphi(q^5)$ in Theorem 2.6, we have

$$\begin{aligned}
\varphi^3(-q)\varphi(-q^5) &= E_{\chi_0, \chi_1}(-q) - 2E_{\chi_0, \chi_1}(q^2) - 4E_{\chi_0, \chi_1}(q^4) \\
(4.1) \quad &\quad + 5E_{\chi_1, \chi_0}(-q) + 10E_{\chi_1, \chi_0}(q^2) - 20E_{\chi_1, \chi_0}(q^4).
\end{aligned}$$

Appealing to Theorem 2.3, we obtain

$$(4.2) \quad E_{\chi_0, \chi_1}(-q) = -E_{\chi_0, \chi_1}(q) - 2E_{\chi_0, \chi_1}(q^2) + 4E_{\chi_0, \chi_1}(q^4),$$

$$(4.3) \quad E_{\chi_1, \chi_0}(-q) = -E_{\chi_1, \chi_0}(q) + 2E_{\chi_1, \chi_0}(q^2) + 4E_{\chi_1, \chi_0}(q^4).$$

Substituting (4.2) and (4.3) in (4.1), we obtain

$$(4.4) \quad \varphi^3(-q)\varphi(-q^5) = -E_{\chi_0, \chi_1}(q) - 4E_{\chi_0, \chi_1}(q^2) - 5E_{\chi_1, \chi_0}(q) + 20E_{\chi_1, \chi_0}(q^2).$$

It can easily be seen that

$$(4.5) \quad -E_{\chi_0, \chi_1}(q) - 4E_{\chi_0, \chi_1}(q^2) = 1 + \sum_{n=1}^{\infty} \left(\sum_{d|n} (-1)^d \left(\frac{5}{d} \right) d \right) q^n,$$

$$(4.6) \quad -E_{\chi_1, \chi_0}(q) + 4E_{\chi_1, \chi_0}(q^2) = \sum_{n=1}^{\infty} \left(\sum_{d|n} (-1)^d \left(\frac{5}{n/d} \right) d \right) q^n.$$

Now, appealing to (1.1) and (4.4)–(4.6), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 1, 5; n)(-q)^n &= \varphi^3(-q)\varphi(-q^5) \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{d|n} (-1)^d \left(\frac{5}{d} \right) d \right) q^n + 5 \sum_{n=1}^{\infty} \left(\sum_{d|n} (-1)^d \left(\frac{5}{n/d} \right) d \right) q^n, \end{aligned}$$

from which we deduce

$$N(1, 1, 1, 5; n) = \sum_{d|n} (-1)^{n+d} \left(\frac{5}{d} \right) d + 5 \sum_{d|n} (-1)^{n+d} \left(\frac{5}{n/d} \right) d,$$

which agrees with known results, see for example [1, Theorem 5.1]. Similarly, one can show that our formula for $N(1, 5, 5, 5; n)$ given in Theorem 3.2 agrees with the result in [1, Theorem 6.1].

Remark 4.2. Appealing to Lemma 2.1 and Theorem 2.3, we obtain the following identities:

$$\begin{aligned} L(q) - 4L(q^4) &= \frac{1}{8} \frac{\eta^{20}(2z)}{\eta^8(z)\eta^8(4z)}, \\ E_{\chi_0, \chi_1}(q) &= -\frac{1}{5} \frac{\eta^5(z)}{\eta(5z)}, \\ E_{\chi_1, \chi_0}(q) &= \frac{\eta^5(5z)}{\eta(z)}, \\ E_{\chi_0, \chi_2}(q) &= -\frac{1}{2} \frac{\eta^2(z)\eta(2z)\eta^3(4z)}{\eta^2(8z)}, \\ E_{\chi_2, \chi_0}(q) &= \frac{\eta^3(2z)\eta(4z)\eta^2(8z)}{\eta^2(z)}, \\ E_{\chi_0, \chi_1}(q) + 4E_{\chi_0, \chi_1}(q^2) &= -\frac{\eta(z)\eta^2(2z)\eta^3(5z)}{\eta^2(10z)}, \\ E_{\chi_1, \chi_0}(q) + E_{\chi_1, \chi_0}(q^2) &= \frac{\eta^3(2z)\eta^2(5z)\eta(10z)}{\eta^2(z)}, \\ E_{\chi_1, \chi_0}(q) - 4E_{\chi_1, \chi_0}(q^2) &= \frac{\eta^3(z)\eta(5z)\eta^2(10z)}{\eta^2(2z)}, \\ E_{\chi_0, \chi_2}(q) - 2E_{\chi_2, \chi_0}(q) &= -\frac{1}{2} \frac{\eta^{13}(4z)}{\eta^2(z)\eta(2z)\eta^6(8z)}, \\ E_{\chi_0, \chi_2}(q) - 4E_{\chi_2, \chi_0}(q) &= -\frac{1}{2} \frac{\eta^{13}(2z)}{\eta^6(z)\eta(4z)\eta^2(8z)}, \\ E_{\chi_0, \chi_1}(q) - 2E_{\chi_0, \chi_1}(q^2) - 4E_{\chi_0, \chi_1}(q^4) &= \frac{\eta^5(2z)\eta^7(10z)}{\eta(z)\eta(4z)\eta^3(5z)\eta^3(20z)}, \end{aligned}$$

$$E_{\chi_1, \chi_0}(q) + 2E_{\chi_1, \chi_0}(q^2) - 4E_{\chi_1, \chi_0}(q^4) = \frac{\eta^7(2z)\eta^5(10z)}{\eta^3(z)\eta^3(4z)\eta(5z)\eta(20z)}.$$

Remark 4.3. Set $a := \varphi(q)$, $b := \varphi(q^2)$, $c := \varphi(q^5)$ and $d := \varphi(q^{10})$. We obtain the following identities from Theorem 2.8:

$$\begin{aligned} ad(-a^2 - b^2 + 5c^2 - 5d^2) + bc(5a^2 - 8b^2 - 5c^2 + 10d^2) &= 12D_1(q), \\ ad(2a^2 - b^2 - 4c^2 + d^2) + bc(-a^2 + b^2 - 5c^2 + 7d^2) &= 24D_2(q), \\ ad(-a^2 + 2b^2 - 7c^2 + 10d^2) + bc(-a^2 + 4b^2 + c^2 - 8d^2) &= 48D_3(q), \\ ad(a^2 - 8b^2 - 5c^2 + 20d^2) + bc(7a^2 - 10b^2 + 5c^2 - 10d^2) &= 48D_4(q). \end{aligned}$$

Remark 4.4. It would be interesting to determine general formulas for the number of representations of a positive integer n by the quaternary quadratic forms with coefficients in $\{1, p, q, pq\}$, where p and q are distinct prime numbers. The case when $p = 2$ and $q = 7$ is treated in [4].

Acknowledgements. The research of the first author was supported by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (RGPIN-418029-2013).

REFERENCES

- [1] A. Alaca, Ş. Alaca and K. S. Williams, On the quaternary forms $x^2 + y^2 + z^2 + 5t^2$, $x^2 + y^2 + 5z^2 + 5t^2$ and $x^2 + 5y^2 + 5z^2 + 5t^2$, *JP J. Algebra Number Theory Appl.* **9** (2007), 37-53.
- [2] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, Nineteen quaternary quadratic forms, *Acta Arith.* **130** (2007), 277-310.
- [3] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, The number of representations of a positive integer by certain quaternary quadratic forms, *Int. J. Number Theory* **5** (2009), 13-40.
- [4] A. Alaca and J. Alanazi, Representations by quaternary quadratic forms with coefficients 1, 2, 7 or 14, *INTEGERS* (2016), in press.
- [5] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer. Math. Soc., Student Mathematical Library, 2006.
- [6] B. Gordon and D. Sinor, Multiplicative properties of η -products, *Springer Lecture Notes in Math.* **1395** (1989), 173-200.
- [7] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, 1829. Reprinted in *Gesammelte Werke* Vol. 1, Chelsea, New York, 1969, 49-239.
- [8] H. D. Kloosterman, On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$, *Proc. London Math. Soc.* **25** (1926), 143-173.
- [9] L. J. P. Kilford, *Modular Forms: A Classical and Computational Introduction*, 2nd edition, Imperial College Press, London, 2015.
- [10] G. Köhler, *Eta Products and Theta Series Identities*, Springer Monographs in Mathematics, Springer, 2011.
- [11] G. Ligozat, Courbes modulaires de genre 1, *Bull. Soc. Math. France* **43** (1975), 5-80.
- [12] J. Liouville, Sur les deux formes $x^2 + y^2 + 2(z^2 + t^2)$, *J. Math. Pures Appl.* **5** (1860), 269-272.
- [13] J. Liouville, Sur les deux formes $x^2 + y^2 + z^2 + 2t^2$ and $x^2 + 2(y^2 + z^2 + t^2)$, *J. Math. Pures Appl.* **6** (1861), 225-230.

- [14] J. Liouville, la forme $x^2 + y^2 + z^2 + 5t^2$, *J. Math. Pures Appl.* **9** (1864), 1-12.
- [15] J. Liouville, la forme $x^2 + 5(y^2 + z^2 + t^2)$, *J. Math. Pures Appl.* **9** (1864), 17-22.
- [16] J. Liouville, la forme $x^2 + y^2 + 5z^2 + 5t^2$, *J. Math. Pures Appl.* **10** (1865), 1-8.
- [17] G. Lomadze, Über die Darstellung der Zahlen durch einige quaternäre quadratische Formen, *Acta Arith.* **5** (1959), 125-170.
- [18] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q -Series*, Am. Math. Soc., 2004.
- [19] W. A. Stein, *Modular Forms, a Computational Approach*, Amer. Math. Soc., Graduate Studies in Mathematics, 2007.
- [20] K. S. Williams, On the representations of a positive integer by the forms $x^2 + y^2 + z^2 + 2t^2$ and $x^2 + 2y^2 + 2z^2 + 2t^2$, *Int. J. Modern Math.* **3** (2008), 225-230.
- [21] K. S. Williams, *Number Theory in the Spirit of Liouville*, Cambridge University Press, 2011.

Ayşe Alaca and Mada Altiary
School of Mathematics and Statistics
Carleton University
Ottawa, ON K1S 5B6, Canada

AyseAlaca@cunet.carleton.ca
MadaAltiary@cmail.carleton.ca